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BOUNDED ANALYTIC MAPS, WALL FRACTIONS AND ABC–FLOW

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Abstract. In this work we study the qualitative properties of real analytic bounded maps defined in the infinite complex strip. The main tool is approximation by continued g–fractions of Wall [12]. As an application, the ABC–flow system is considered which is essential to the origin of the solar magnetic field [1].

1. Introduction

In 1948 Hubert Wall introduced the particular class of functional continued fractions called g–fractions. The objective of the present study is to broaden our understanding of Wall's ideas in the dynamical system theory.

In this section we will remind the reader of some key facts from the analytic theory of continued fractions. Let $\mathbb{H} = \mathbb{C}_- \cup \mathbb{C}_+ \cup (-1, +\infty)$ where $\mathbb{C}_+ = \{z \in \mathbb{C} : Im(z) > 0\}$, $\mathbb{C}_- = \{z \in \mathbb{C} : Im(z) < 0\}$.

For arbitrary real sequence $g_i \in [0, 1]$, $i \geq 1$ we call *g–fraction* the continued fraction

$$g(z) = \{g_1, g_2, ...|z\} = \frac{1}{1} {}_{+} \frac{g_1 z}{1} {}_{+} \frac{(1-g_1)g_2 z}{1} {}_{+} \frac{(1-g_2)g_3 z}{1} \cdots , \tag{1.1}$$

converging uniformly on compact sets of $\mathbb{H}$ to an analytic function (see [12]). The map $g(z)$ is rational if and only if $g_k \in \{0, 1\}$, for some $k \geq 1$.

In particular, if $g_i = p \in (0, 1)$, $i \geq 1$ then $g(z)$ is algebraic and is given explicitly by

$$\{p, p, \cdots \mid z\} = \frac{2(1-p)}{1 - 2p + \sqrt{1 + 4p(1-p)z}}, \quad z \in \mathbb{C}_- \cup \mathbb{C}_+ \cup (-1/4p(1-p), +\infty) \,. \tag{1.2}$$

It is known that some ratios of hypergeometric functions can be expressed with help of g–fractions. Let a, b, c are real constants satisfying $-1 \leq a \leq c$, $0 \leq b \leq c \neq 0$ and $F(a, b, c, z)$ be the hypergeometric function of Gauss. Then, as shown in [6]:

$$\frac{F(a+1, b, c, -z)}{F(a, b, c, -z)} = \{g_1, g_2, ...|z\}, \quad z \in \mathbb{H} \,, \tag{1.3}$$

where

$$g_{2k} = \frac{a+k}{c+2k-1}, \quad g_{2k-1} = \frac{b+k-1}{c+2k-2}, \quad k \geq 1 \,. \tag{1.4}$$

2000 *Mathematics Subject Classification.* 37C30, 30E05, 11J70.

Key words and phrases. Continued fractions, real analytic maps, dynamical systems, ABC flow, Navier–Stokes equations, fast dynamo.

In the simplest case:

$$\frac{1}{z}\ln(1+z) = \frac{F(1,1,2,-z)}{F(0,1,2,-z)}\,. \tag{1.5}$$

We define the truncated continued g–fraction as the n–order approximation of (1.1):

$$\{g_1, g_2, ..., g_n|z\} = \frac{1}{1}\,{}_{+}\,\frac{g_1 z}{1}\,{}_{+}\,\frac{(1-g_1)g_2 z}{1}\,\dots\,\frac{(1-g_{n-1})g_n z}{1}\,, \tag{1.6}$$

which is a rational function of z analytic in $\mathbb{H}$.

The next result is due to Gragg [5]:

Theorem 1.1. *Let $z \in (-1,+\infty)$, then the n–order truncation error satisfies:*

$$|\{g_1, g_2, ...|z\} - \{g_1, g_2, ..., g_n|z\}| \le \left|1 - \frac{1}{1+z}\right| \left|\frac{1-\sqrt{1+z}}{1+\sqrt{1+z}}\right|^n\,, \tag{1.7}$$

and does not depend on values of g_i, $i \ge 1$.

The real *a priori* bounds for the g–fraction are given by the next result:

Theorem 1.2. ([9])
a) Let $k = 2n+1$, $n = 0, 1, ...,$ then

$$A_k(z) \le g(z) \le B_k(z), \quad -1 < z < +\infty\,, \tag{1.8}$$

where

$$A_k(z) = \{g_1, g_2, ..., g_k|z\}, \quad B_k(z) = \{g_1, g_2, ..., g_k, 1|z\}\,. \tag{1.9}$$

b) Let $k = 2n$, $n = 1, 2, ...,$ then

$$A^+_k(z) \le g(z) \le B^+_k(z), \quad 0 \le z < +\infty, \tag{1.10}$$

$$A^-_k(z) \le g(z) \le B^-_k(z), \quad -1 < z < 0, \tag{1.11}$$

where

$$A^+_k(z) = \{g_1, g_2, ..., g_k, 1|z\}, \quad B^+_k = \{g_1, g_2, ..., g_k|z\}\,, \tag{1.12}$$

and $A^-_k = B^+_k$, $B^-_k = A^+_k$.

Using the above formulas we write below the rational *a priori* bounds for the g–fraction (1.1) corresponding to $k = 1, 2, 3$:
Case $k = 1$.

$$A_1(z) = \frac{1}{1+g_1 z}, \quad B_1(z) = \frac{1+(1-g_1)z}{1+z}\,. \tag{1.13}$$

Case $k = 2$.

$$A_2^+(z) = \frac{(1-g_1g_2)z+1}{(1+z)(g_1(1-g_2)z+1)}, \quad B_2^+(z) = \frac{g_2(1-g_1)z+1}{(g_1-g_1g_2+g_2)z+1}\,, \tag{1.14}$$

$$A_2^- = B_2^+, B_2^- = A_2^+\,. \tag{1.15}$$

Case $k=3$.

$$A_3(z) = \frac{(g_3+g_2-g_3g_2-g_2g_1)z+1}{g_1g_3(1-g_2)z^2+(g_3+g_2+g_1-g_3g_2-g_1g_2)z+1}\,. \tag{1.16}$$

$$B_3(z) = \frac{g_2(1-g_3)(1-g_1)z^2+(1+g_2-g_3g_2-g_1g_2)z+1}{(1+z)((g_1+g_2-g_3g_2-g_1g_2)z+1)}\,. \tag{1.17}$$

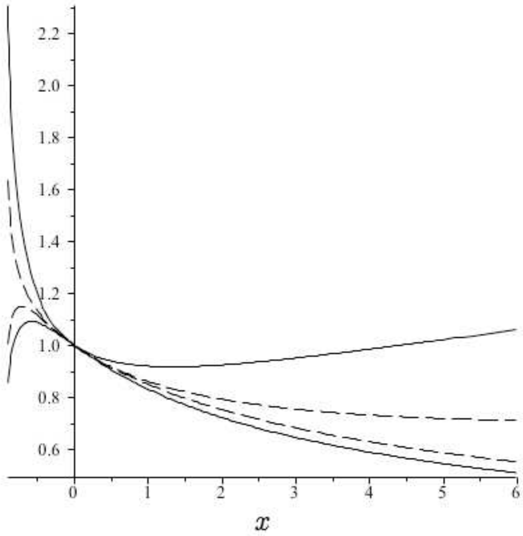

FIGURE 1. Bounds $r(x)A_1(x)$, $r(x)B_1(x)$ (bold line) and $r(x)A_2^{\pm}(x)$, $r(x)B_2^{\pm}(x)$ (dashed line) for $x \in (-0.9, 6)$, $g_1 = 0.7$, $g_2 = 0.3$, $r(x) = \sqrt{1+x}$.

The interesting link between g–fractions and probability theory was reported by Gerl [4]. One considers the nearest-neighbour random walks X_n, $n = 0, 1, 2, ...$ on $\mathbb{N}_0 = \{0, 1, 2, ...\}$ with the one-step transition probabilities $p_{i,k} = \mathrm{Prob}[X_{n+1} = k \mid X_n = i]$ defined by

$$p_{0,1} = 1, \quad p_{j,j-1} = g_j, \quad p_{j,j+1} = 1 - g_j \tag{1.18}$$

with $0 < g_j < 1$, $j \geq 1$ and $p_{j,k} = 0$ in any other case.

We introduce $p_{0,0}^{2n} = \mathrm{Prob}[X_{2n} = 0 \mid X_0 = 0]$– the probability of return to 0 in $2n$ steps. The generating function for this sequence

$$G_0(z) = \sum_{n=0}^{\infty} (-1)^n \, p_{0,0}^{2n} \, z^n, \tag{1.19}$$

can be written then as a g–fraction:

$$G_0(z) = \{g_1, g_2, ...|z\}\,. \tag{1.20}$$

In the next section we will examine the relaltion between g–fractions and real analytic bounded functions.

2. Functions bounded in the complex strip

We denote $\mathbb{A}_{M,B}$ the set of functions $f(z)$ satisfying the following conditions

a) $f(z)$ is holomorphic in the infinite strip $S_B = \{z \in \mathbb{C} : |Im(z)| < B\}$, $B > 0$.

b) $f(\mathbb{R}) \subset \mathbb{R}$.

c) $|f(z)| < M$, $M > 0$, $\forall\ z \in S_B$.

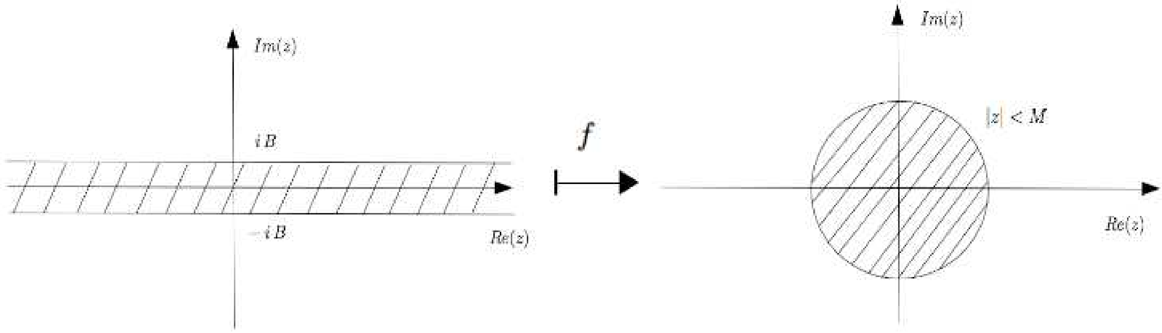

Figure 2. Map defined by $f(z) \in \mathbb{A}_{M,B}$.

Our present aim is to obtain g–fraction representation for the class $\mathbb{A}_{M,B}$.

Theorem 2.1. *Let $f(z) \in \mathbb{A}_{M,B}$. Then for some $\mu_0 > 0$ and $g_k \in [0,1]$, $k \geq 1$ one has*

$$f(z) = M\left(1 - \frac{2}{\mu_0 \exp\left(\frac{\pi z}{2B}\right)\{g_1, g_2, ...|\exp\left(\frac{\pi z}{B}\right) - 1\} + 1}\right). \tag{2.1}$$

Proof. We introduce the complex domains

$$\mathbb{D}_M = \{z \in \mathbb{C} : |z| < M\}, \quad \mathbb{H}_+ = \{z \in \mathbb{C} : Re(z) > 0\}\,, \tag{2.2}$$

and the conformal maps defined by:

$$m(z) = \frac{M+z}{M-z}, \quad m : \mathbb{D}_M \to \mathbb{H}_+\,, \tag{2.3}$$

and

$$l(z) = \frac{B}{\pi}\log(1+z), \quad l : \mathbb{H} \to S_B\,, \tag{2.4}$$

$$\eta(z) = l^{-1}(z) = \exp\left(\frac{\pi z}{B}\right) - 1\,. \tag{2.5}$$

We note that η is a bijection between $\mathbb{R}$ and $(-1,+\infty)$.

One verifies that the composition $F = m \circ f \circ l$ is holomorphic in $\mathbb{H}$ and $F(\mathbb{H}) \subset \mathbb{H}_+$ with $F(z) \in \mathbb{R}$ for $z > -1$. Thus, according to theorem of Wall [12], p. 279 F can be written as follows

$$F(z) = \mu_0 \sqrt{1+z} \int_0^1 \frac{d\mu(u)}{1+zu}\,, \tag{2.6}$$

for some nondecreasing real bounded function $\mu(u)$, $u \in (0,1)$ and $\mu_0 > 0$.

For $f = m^{-1} \circ F \circ \eta$ one obtains the following formula

$$f(z) = M\left(1 - \frac{2}{\mu_0 \exp\left(\frac{\pi z}{2B}\right) \int_0^1 \frac{d\mu(u)}{1+(\exp\left(\frac{\pi z}{B}\right)-1)u} + 1}\right). \tag{2.7}$$

The integral in (2.7) can be transformed to the continued g–fraction form [12]

$$\int_0^1 \frac{d\mu(u)}{1+(\exp\left(\frac{\pi z}{B}\right)-1)u} = \left\{g_1, g_2, ...\middle|\, \exp\left(\frac{\pi z}{B}\right) - 1\right\}, \quad \text{for some} \quad g_k \in [0,1]\,, \tag{2.8}$$

that together with (2.7) implies (2.1). Proof is finished. □

Let

$$\theta(z) = \frac{1}{M} f(Bz/\pi)\,. \tag{2.9}$$

To calculate the coefficients g_p in (2.1) one has formulas:

$$g_p = C_p(\theta(0), \theta'(0), ..., \theta^{(p)}(0)), \quad p \geq 1\,, \tag{2.10}$$

with rational functions C_p determined by calculation of derivatives of both sides of (2.1) at $z = 0$. The recurrent formulas for all C_p can be derived from [12], p. 203.

Introducing

$$\theta_n = \theta^{(n)}(0) = \frac{1}{M}\frac{B^n}{\pi^n} f^{(n)}(0), \quad n \geq 0\,, \tag{2.11}$$

we provide below explicit formulas for μ_0, g_1, g_2:

$$\mu_0 = \frac{1+\theta_0}{1-\theta_0}\,, \tag{2.12}$$

$$g_1 = \frac{1}{2}\frac{1-4\theta_1-\theta_0^2}{1-\theta_0^2}\,, \tag{2.13}$$

$$g_2 = \frac{1}{2}\frac{(16\theta_1^2 - 8\theta_2 - \theta_0 + \theta_0^2 + \theta_0^3 - 8\theta_2\theta_0 - 1)(1-\theta_0)}{(1-\theta_0^2+4\theta_1)(4\theta_1 - 1 + \theta_0^2)}\,. \tag{2.14}$$

Definition 2.1. *We denote by* $\mathbb{A}_{M,B}^{(k)} \subset \mathbb{A}_{M,B}$, $k = 1, 2, ...$ *the set of functions for which the g–fraction representation* (2.1) *satisfies the condition*

$$g_i \notin \{0, 1\}, \quad \forall\, i = 1, \ldots, k\,. \tag{2.15}$$

Our aim is to estimate the time of return of $f(z) \in \mathbb{A}_{M,B}$ to the initial value $f(0)$ i.e to study the points $z_0 \in \mathbb{R}^*$ such that $f(z_0) = f(0)$. To do this we will use the *a priori* bounds (1.13) applied to the g–fraction in formula (2.1). For $p = 2k+1$ one obtains:

$$\left(1 - \frac{2}{\mu_0\sqrt{1+\eta(z)}A_p(\eta(z))+1}\right) \le \frac{f(z)}{M} \le \left(1 - \frac{2}{\mu_0\sqrt{1+\eta(z)}B_p(\eta(z))+1}\right), \quad (2.16)$$

for $z \in \mathbb{R}$.

If $p = 2k$ then

$$\left(1 - \frac{2}{\mu_0\sqrt{1+\eta(z)}A_p^+(\eta(z))+1}\right) \le \frac{f(z)}{M} \le \left(1 - \frac{2}{\mu_0\sqrt{1+\eta(z)}B_p^+(\eta(z))+1}\right), \quad (2.17)$$

for $z \in (0, +\infty)$, and

$$\left(1 - \frac{2}{\mu_0\sqrt{1+\eta(z)}A_p^-(\eta(z))+1}\right) \le \frac{f(z)}{M} \le \left(1 - \frac{2}{\mu_0\sqrt{1+\eta(z)}B_p^-(\eta(z))+1}\right), \quad (2.18)$$

for $z \in (-\infty, 0]$.

In the next theorem, for a given $f \in \mathbb{A}_{M,B}^{(1)}$, we will describe a neighborhood of origin in which $z = 0$ is the only solution of $f(z) = f(0)$.

Theorem 2.2. *Let $f(z) \in \mathbb{A}_{M,B}^{(1)}$, $f'(0) \neq 0$ where g_1 is defined by* (2.11) *and* (2.13) *as function of M, B, $f(0)$, $f'(0)$. Let $\tau \in \mathbb{R}^*$ be the point such that $f(\tau) = f(0)$. Then*

$$|\tau| \ge \frac{2B}{\pi}\left|\log\left(\frac{1-g_1}{g_1}\right)\right| > 0\,. \quad (2.19)$$

Proof. One considers (2.16) with $p = 1$. We have $A_1(0) = B_1(0) = 1$ and define t_1, t_2 as non-zero solutions of the following algebraic equations

$$\sqrt{1+t_1}A_1(t_1) = 1, \quad \sqrt{1+t_2}B_1(t_2) = 1, \quad t_1, t_2 \in (-1, +\infty)\,. \quad (2.20)$$

Simple algebraic calculations show that the only solutions satisfying (2.20) are given by

$$t_1 = \frac{1-2g_1}{g_1^2}, \quad t_2 = \frac{2g_1 - 1}{(1-g_1)^2}\,, \quad (2.21)$$

which are related by

$$\frac{1}{t_1} + \frac{1}{t_2} = -1\,. \quad (2.22)$$

Since $\eta(z)$ is a bijection between $\mathbb{R}$ and $(-1,+\infty)$ there exist unique real numbers T_1, $T_2 \in \mathbb{R}$ satisfying the following equations:

$$\eta(T_1) = t_1, \quad \eta(T_2) = t_2\,. \tag{2.23}$$

As easily seen from (2.5): $T_1 = -T_2$ and the proof of (2.19) follows straightforwardly from the formula (2.4). □

The next result shows that $f(z) \in \mathbb{A}^{(2)}_{M,B}$, under some conditions on derivatives $f^{(p)}(0)$, $p = 0,1,2$, always returns to the initial value $f(0)$ i.e admits the oscillatory property.

Theorem 2.3. *Let $f(z) \in \mathbb{A}^{(2)}_{M,B}$, $f'(0) \neq 0$ where g_1, g_2 are defined by formulas* (2.11) *and* (2.13), (2.14).
I. We assume that the point $(g_1, g_2) \in (0,1)^2$ belongs to one of the four regions E, F, G, H defined by:

$$E = \left\{(g_1, g_2) \in (0,1)^2 \,:\, D_2 \geq 0, 0 < g_1 < 1/2, 0 < g_2 < 1/2\right\}, \tag{2.24}$$

$$F = \left\{(g_1, g_2) \in (0,1)^2 \,:\, D_2 \geq 0, 0 < g_1 < 1/2, 1/2 < g_2 < 1\right\}, \tag{2.25}$$

$$G = \left\{(g_1, g_2) \in (0,1)^2 \,:\, D_1 \geq 0, 1/2 < g_1 < 1, 0 < g_2 < 1/2\right\}, \tag{2.26}$$

$$H = \left\{(g_1, g_2) \in (0,1)^2 \,:\, D_1 \geq 0, 1/2 < g_1 < 1, 1/2 < g_2 < 1\right\}, \tag{2.27}$$

$$D_1 = (1-g_1)^2 - 4g_1^2 g_2(1-g_2),\; D_2 = g_1^2 - 4(1-g_1)^2(1-g_2)g_2\,. \tag{2.28}$$

Let

$$\zeta = \frac{2B}{\pi}\log(t_1^{(2)}) > 0 \quad \text{if} \quad (g_1, g_2) \in E \tag{2.29}$$

$$\zeta = \frac{2B}{\pi}\log(t_2^{(2)}) < 0 \quad \text{if} \quad (g_1, g_2) \in F \tag{2.30}$$

$$\zeta = \frac{2B}{\pi}\log(t_2^{(1)}) < 0 \quad \text{if} \quad (g_1, g_2) \in G \tag{2.31}$$

$$\zeta = \frac{2B}{\pi}\log(t_1^{(1)}) > 0 \quad \text{if} \quad (g_1, g_2) \in H \tag{2.32}$$

where $t_j^{(i)}$, $i,j = 1,2$ are defined as functions of g_1, g_2 by

$$t_{2,1}^{(1)} = \frac{1 - g_1 \pm \sqrt{D_1}}{2g_1(1-g_2)}, \quad t_{2,1}^{(2)} = \frac{g_1 \pm \sqrt{D_2}}{2(1-g_1)g_2}\,. \tag{2.33}$$

Then there exists $\tau \in \mathbb{R}^*$ such that

$$f(\tau) = f(0)\,. \tag{2.34}$$

and

$$\tau \in (0, \zeta) \quad \text{if} \quad \zeta > 0 \quad \text{and} \quad \tau \in (\zeta, 0) \quad \text{if} \quad \zeta < 0\,. \tag{2.35}$$

II. Let $\theta \in \mathbb{R}^*$ be such that $f(\theta) = f(0)$, then

$$\theta \in (-\infty, \frac{2B}{\pi}\log(t_1^{(1)})] \cup [\frac{2B}{\pi}\log(t_2^{(1)}), +\infty), \quad if \quad f'(0) > 0, \tag{2.36}$$

and

$$\theta \in (-\infty, \frac{2B}{\pi}\log(t_1^{(2)})] \cup [\frac{2B}{\pi}\log(t_2^{(2)}), +\infty), \quad if \quad f'(0) < 0\,. \tag{2.37}$$

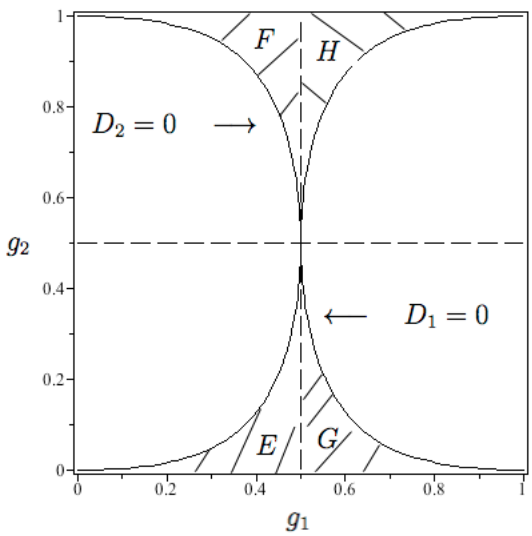

FIGURE 3. Four domains E, F, G, H in the parameter space $(g_1, g_2) \in [0,1]^2$.

Proof. We consider (2.17) with $p = 2$ and define the following real algebraic equations

$$\sqrt{1+x}\,A_2^+(x) = 1\,, \quad x \in (0,+\infty)\,, \quad (\mathrm{A}_1)\,,$$
$$\sqrt{1+x}\,B_2^+(x) = 1\,, \quad x \in (0,+\infty)\,, \quad (\mathrm{B}_1)\,,$$
$$\sqrt{1+x}\,B_2^-(x) = 1\,, \quad x \in (-1,0)\,, \quad (\tilde{\mathrm{A}}_1)\,,$$
$$\sqrt{1+x}\,A_2^-(x) = 1\,, \quad x \in (-1,0)\,. \quad (\tilde{\mathrm{B}}_1)\,,$$

where $A_2^- = B_2^+$, $B_2^- = A_2^+$.

Making the change of variables

$$x = -1 + t^2, \quad t \in \mathbb{R}\,, \tag{2.38}$$

after some elementary transformations, it is easy to show that equations (A_1), (B_1) are equivalent respectively to quadratic equations (A_2) and (B_2) given below

$$P_1(t) = g_1(1-g_2)t^2 - (1-g_1)t + g_1g_2 = 0, \quad t \in \mathbb{R}, \quad (\mathrm{A}_2)$$
$$P_2(t) = g_2(1-g_1)t^2 - g_1t + (1-g_1)(1-g_2) = 0, \quad t \in \mathbb{R}\,. \quad (\mathrm{B}_2)$$

Remark 2.1. *We notice that $P_2(t)$ is obtained from $P_1(t)$ by transformation*

$$g_i \mapsto 1 - g_i, \quad i = 1, 2\,. \tag{2.39}$$

The polynomial $P_1(t) = 0$ has two real roots $t_1^{(1)}, t_2^{(1)} \in \mathbb{R}$

$$t_1^{(1)} = \frac{1 - g_1 - \sqrt{D_1}}{2g_1(1 - g_2)}, \quad t_2^{(1)} = \frac{1 - g_1 + \sqrt{D_1}}{2g_1(1 - g_2)}, \quad t_1^{(1)} \le t_2^{(1)}\,, \tag{2.40}$$

if and only if the following condition holds

$$D_1 = (1 - g_1)^2 - 4g_1^2 g_2 (1 - g_2) \ge 0\,. \tag{2.41}$$

$P_2(t) = 0$ has two real solutions $t_1^{(2)}, t_2^{(2)} \in \mathbb{R}$

$$t_1^{(2)} = \frac{g_1 - \sqrt{D_2}}{2(1 - g_1)g_2}, \quad t_2^{(2)} = \frac{g_1 + \sqrt{D_2}}{2(1 - g_1)g_2}, \quad t_1^{(2)} \le t_2^{(2)}\,, \tag{2.42}$$

if and only if

$$D_2 = g_1^2 - 4(1 - g_1)^2 (1 - g_2) g_2 \ge 0\,. \tag{2.43}$$

Applying the Vieta's formulas to polynomials A_2 and B_2, and taking into account that $g_i \in (0, 1)$, $i = 1, 2$ one checks that:

$$t_j^{(i)} > 0, \quad i, j = 1, 2\,. \tag{2.44}$$

Case A. Let $f'(0) > 0 (\Leftrightarrow g_1 < 1/2)$. Then $f(z)$ is increasing function in the interval $(-\epsilon, \epsilon)$ for some small $\epsilon > 0$. We assume that inequality $D_2 \ge 0$ holds, so both roots $t_1^{(2)}$ and $t_2^{(2)}$ are real. One has $P_2(1) = 1 - 2g_1 > 0$, so, in view of (2.44), either $0 < t_1^{(2)} \le t_2^{(2)} < 1$ (a) or $1 < t_1^{(2)} \le t_2^{(2)}$ (b). One verifies with help of (2.42) that (a) is equivalent to $L_2 = g_1 - 2(1 - g_1)g_2 < 0$ and (b) to $L_2 > 0$. Thus, in view of (2.38), if (b) holds, the equation (B_1) will have solution $x = -1 + {t_1^{(2)}}^2 \in (0, +\infty)$ and if (a) holds, $(\tilde{B}_1)$ will have solution $x = -1 + {t_2^{(2)}}^2 \in (-1, 0)$ in view of (2.38).

One verifies directly that the condition $0 < g_1 < 1/2$ implies $D_1 > 0$. So, the the both roots $t_1^{(1)}$ and $t_2^{(1)}$ are real distinct numbers. Since $P_1(1) = 2g_1 - 1 < 0$ we have $0 < t_1^{(1)} < 1 < t_2^{(1)}$. So, the equation (A_1) will have the unique real solution $y_2 = -1 + {t_2^{(1)}}^2 \in (0, +\infty)$ and $(\tilde{A}_1)$ will have the unique real solution $y_1 = -1 + {t_1^{(1)}}^2 \in (-1, 0)$.

Case B. Let $f'(0) < 0 (\Leftrightarrow g_1 > 1/2)$. Then $f(z)$ is decreasing function in the interval $(-\epsilon, \epsilon)$ for some small $\epsilon > 0$. We assume that inequality $D_1 \ge 0$ holds, so both roots $t_1^{(1)}$ and $t_2^{(1)}$ are real. One has $P_1(1) = 2g_1 - 1 > 0$, so, in view of (2.44), either $0 < t_1^{(1)} \le t_2^{(1)} < 1$ (c) or $1 < t_1^{(1)} \le t_2^{(1)}$ (d). One verifies with help of (2.33) that (c) is equivalent to $L_1 = 1 - g_1 - 2g_1(1 - g_2) < 0$ and (d) to $L_2 = 1 - g_1 - 2g_1(1 - g_2) > 0$. Thus, in

view of (2.38), if (d) holds, the equation (A_1) will have solution $x = -1 + {t_1^{(1)}}^2 \in (0, +\infty)$ and if (c) holds, $(\tilde{A}_1)$ will have solution $x = -1 + {t_2^{(1)}}^2 \in (-1, 0)$ in view of (2.38).

One verifies directly that the condition $g_1 > 1/2$ implies $D_2 > 0$. So, the the both roots $t_1^{(2)}$ and $t_2^{(2)}$ are real distinct positive numbers. Since $P_2(1) = 1 - 2g_1 < 0$ we have $0 < t_1^{(2)} < 1 < t_2^{(2)}$. So, the equation (B_1) will have the unique real solution given by $y_2 = -1 + {t_2^{(2)}}^2 \in (0, +\infty)$ and $(\tilde{B}_1)$ will have the unique real solution defined by $y_1 = -1 + {t_1^{(2)}}^2 \in (-1, 0)$.

Since $\eta(z)$ is a bijection of $\mathbb{R}$ and $(-1, +\infty)$, there exists unique real number $\zeta \in \mathbb{R}$ satisfying equation $\eta(\zeta) = x$ with $x \in (-1, +\infty)$ defined above. Then, as follows from (2.17), (2.18), there exists τ satisfying (2.34) if one of the cases (2.24)-(2.27) holds. One has $\tau \in (0, \zeta)$ if $\zeta > 0$ and $\tau \in (\zeta, 0)$ if $\zeta < 0$.

Using $y_{1,2}$ defined above, we define $z_1 < 0$ and $z_2 > 0$ as unique real solutions of $\eta(z_i) = y_i$, $i = 1, 2$. Let now $\theta \in \mathbb{R}^*$ be such that $f(\theta) = f(0)$, then $\theta \in (-\infty, z_1] \cup [z_2, +\infty)$ that shows (2.36),(2.37) and finishes the proof.

□

3. Applications to solutions of the ABC–flow equations

The ABC–flow is a system of three ordinary differential equations

$$\frac{dx_1}{dz} = A \sin x_3 + C \cos x_2, \quad \frac{dx_2}{dz} = B \sin x_1 + A \cos x_3, \quad \frac{dx_3}{dz} = C \sin x_2 + B \cos x_1\,, \tag{3.1}$$

depending on three arbitrary real positive constants $(A, B, C) \neq (0, 0, 0)$. This vector field appears as an exact solution of the Euler equation without forcing (see [1] for details). It is essential to the origin of magnetic fields in large astrophysical bodies like the Earth, the Sun and galaxies.

We define

$$\delta = \max\{A + C, B + A, C + B\} > 0\,. \tag{3.2}$$

Since the vector field of (3.1) is a bounded one it is complete in $\mathbb{R}^3$ and hence all its real solutions $(x_1(z), x_2(z), x_3(z))$ are defined for $z \in \mathbb{R}$. Moreover, the following result holds

Lemma 3.1. *Let $\epsilon > 0$ be an arbitrary positive real number. Then every real solution $(x_1(z), x_2(z), x_3(z))$ of (3.1) is analytic function of z in the complex infinite strip*

$$S_b = \{z \in \mathbb{C} \,:\, |Im(z)| < b\}, \quad b = \frac{1}{\delta}\,\frac{\epsilon}{\mathrm{ch}(\epsilon)}\,. \tag{3.3}$$

Proof. For a given $\epsilon > 0$ and a real triplet $(X_1, X_2, X_3) \in \mathbb{R}^3$, we introduce the complex domain $\mathcal{D}^{\epsilon}_{X_1,X_2,X_3} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \,:\, |x_i - X_i| < \epsilon,\, i = 1, 2, 3\}$. Writing the system (3.1) in the form $\dot{x}_i = f_i(x_1, x_2, x_3)$, $i = 1, 2, 3$ one easy verifies that

$$|f_i(P)| < \delta\, \mathrm{ch}(\epsilon), \quad \forall\, P = (x_1, x_2, x_3) \in \mathcal{D}^{\epsilon}_{X_1,X_2,X_3}, \quad 1 \le i \le 3\,, \tag{3.4}$$

in view of the following elementary property:

Proposition 3.1. *The trigonometric functions* $\cos x$, $\sin x$ *are analytic and bounded in absolute value by* $\mathrm{ch}(\epsilon)$ *in the complex disk* $|x - x_0| < \epsilon$ *with a center* $x_0 \in \mathbb{R}$.

We can apply now the classical theorem of Picard (see for example [3]) from the analytic theory of ordinary differential equations:

Theorem 3.1. *Let* $Q_j(q_1, q_2, \dots, q_n)$, $1 \le j \le n$ *be analytic functions in the complex domain*

$$|q_i - \bar{q}_i| < q'_i, \quad 1 \le i \le n\,, \tag{3.5}$$

for some $q'_i > 0$, $\bar{q}_i \in \mathbb{C}$, $1 \le i \le n$. *We assume there exist positive constants* $Q'_j > 0$, $1 \le j \le n$ *such that*

$$|Q_j(q_1, q_2, \dots, q_n)| < Q'_j, \quad 1 \le j \le n\,, \tag{3.6}$$

if the conditions (3.5) *hold.*

Then the system of n *ordinary equations*

$$\frac{dq_j}{dz} = Q_j(q_1, q_2, \dots, q_n), \quad 1 \le j \le n\,, \tag{3.7}$$

admits the unique solution analytic in the complex disc

$$D \,:\, |z| < T', \quad T' = \min\left\{\frac{q'_1}{Q'_1}, \frac{q'_2}{Q'_2}, \dots, \frac{q'_n}{Q'_n}\right\}\,, \tag{3.8}$$

satisfying the initial conditions $q_i(0) = \bar{q}_i$, $1 \le i \le n$.

Moreover,

$$|q_i(z) - \bar{q}_i| < q'_i, \quad z \in D, \quad 1 \le i \le n\,. \tag{3.9}$$

According to this theorem the solution of (3.1) $(x_1(z), x_2(z), x_3(z))$ defined by the initial condition $(x_1(z_0), x_2(z_0), x_3(z_0)) = (X_1, X_2, X_3)$, $z_0 \in \mathbb{R}$ will be analytic in the disk $D_{z_0,T'} \,:\, |z - z_0| < T'$ of the complex time plane with $T' = \dfrac{\epsilon}{\delta\, \mathrm{ch}(\epsilon)}$. Since the system (3.1) is autonomous and $S_b = \cup_{z_0 \in \mathbb{R}} D_{z_0,T'}$ that finishes the proof. ☐

The solutions of the ABC–flow are defined on the torus $\mathbb{T}^3 = \{(x_1, x_2, x_3) \,:\, x_i \bmod 2\pi\}$. Thus, instead of $x_i(z)$ it is natural to consider the projections $\sin(x_i(z)) \,:\, [0, 2\pi] \to [0, 1]$.

Lemma 3.2. *Let* $(x_1(z), x_2(z), x_3(z))$ *be any real solution of the ABC–flow* (3.1)*. Then all functions* $\phi_i(z) = \sin(x_i(z))$*,* $i = 1, 2, 3$ *are analytic in* S_b *defined by* (3.3) *and bounded in absolute value by* $M = \mathrm{ch}(\epsilon)$*.*

Proof. Let $x_j(z) \in (x_1(z), x_2(z), x_3(z))$. Then, $\phi_j(z) = \sin(x_j(z))$ is analytic in S_b as a composition of analytic maps. Let $z_0 \in \mathbb{R}$, then for the complex disc $D_{z_0,b} : |z - z_0| < b$ we have obviously $D_{z_0,b} \subset S_b$. According to (3.9) $\forall\, z \in D_{z_0,b} : |x_j(z) - x_j(z_0)| < \epsilon$. That completes the proof in view of $S_b = \cup_{z_0 \in \mathbb{R}} D_{z_0,b}$ and Proposition 3.1.

□

The function $e(\epsilon) = \epsilon/\mathrm{ch}(\epsilon)$ reaches the unique maximal value for $\epsilon \in \mathbb{R}_+$ which we denote e_{max}. The direct computation gives $e_{max} = e(\epsilon_{max}) = 0.6627$ for $\epsilon_{max} = 1.1997$.

We consider an arbitrary solution $\Gamma_{\alpha,\beta}$ of (3.1) starting from the plane $x_1 = 0$, and defined by initial conditions of the form

$$x_1(0) = 0, \quad x_2(0) = \alpha, \quad x_3(0) = \beta, \quad \alpha, \beta \in [0, 2\pi]\,. \tag{3.10}$$

Let $f(z) = \phi_1(z) = \sin(x_1(z))$. We have $f(0) = 0$, $f \in \mathbb{A}_{M,b}$ and the formulas (2.11), (2.13), (2.14) give

$$\theta_0 = 0, \quad \theta_1 = \frac{1}{\pi\delta}\,\frac{\epsilon}{\mathrm{ch}^2(\epsilon)}(A\sin\beta + C\cos\alpha), \quad \theta_2 = \frac{1}{\pi^2\delta^2}\,\frac{\epsilon^2}{\mathrm{ch}^3(\epsilon)}AB\cos\beta\,, \tag{3.11}$$

and

$$g_1 = \frac{1}{2}\left(1 - 4\theta_1\right), \quad g_2 = \frac{1}{2}\left(1 + \frac{8\theta_2}{1 - 16\theta_1^2}\right)\,. \tag{3.12}$$

Since $f \in \mathbb{A}_{M,b}$, according to Theorem 2.1 we have $g_{1,2} \in [0, 1]$. Indeed, one easy verifies that the strict inequalities always hold: $0 < g_{1,2} < 1$, so $f \in \mathbb{A}^{(2)}_{M,B}$.

Let $\tau \neq 0$ be such that $f(\tau) = \sin(x_1(\tau)) = 0$. According to Theorem 2.2 we have the following lower bound

$$|\tau| \geq \frac{4\,\epsilon}{\pi\,\delta\,\mathrm{ch}(\epsilon)}\mathrm{arctanh}\left(\frac{4\,\epsilon}{\pi\,\delta\,\mathrm{ch}^2(\epsilon)}(A\sin\beta + C\cos\alpha)\right)\,, \tag{3.13}$$

which holds for any $(\alpha, \beta) \in [0, 2\pi]^2$ and arbitrary $\epsilon > 0$, δ is defined by (3.2). This means that the solution $\Gamma_{\alpha,\beta}$ starting from the plane $x_1 = 0$ at $z = 0$ can not cross any of the planes $x_1 = 0, \pm\pi$ earlier than permitted by (3.13). In practice, one can use a freedom in the chose of ϵ in order to make (3.13) optimal. The more precise lower bounds, involving g_2 i.e the second derivative of f at 0, are given by (2.36), (2.37).

Now we will analyze the upper bounds given by Theorem 2.3.

The conditions (2.24), (2.27), defining the domaines E and F in the parameter space $(\alpha, \beta) \in [0, 2\pi]^2$, are respectively equivalent to

$$\{\theta_1 > 0, \quad \theta_2 \leq -\frac{\sqrt{\theta_1}}{2}(1 - 4\theta_1)\} \quad (E) \quad \text{and} \quad \{\theta_1 < 0, \quad \theta_2 \geq \frac{\sqrt{|\theta_1|}}{2}(1 + 4\theta_1)\} \quad (H) \tag{3.14}$$

(the similar descriptions can be derived for the domaines F and G). One can show that for any positive constants A, B, C and ϵ the above sets of parameter values (α, β) are non empty in $[0, 2\pi]^2$. Indeed, it is sufficient to put $\alpha = \pi/2$ in (3.11) and consider the values $(\beta \to \pi,\ \beta < \pi)$ and $(\beta \to 2\pi,\ \beta < 2\pi)$ respectively to satisfy (E) and (H).

The inequalities (3.14) define those values of α, β for which we can guarantee the existence of $\tau > 0$ such that $\sin(x_1(\tau)) = 0$. From the dynamical point of view it corresponds to returning of the trajectory $\Gamma_{\alpha,\beta}$ either to the initial plane $x_1 = 0$ or to its intersection with planes $x_1 = \pm\pi$.

The corresponding upper bounds for τ can be calculated with help of formulas (2.24), (2.27) and (2.33).

4. Conclusions and numerical results

We consider the particular case of the ABC–flow (3.1) defined by the parameter values $(A, B, C) = (1 \cdot 10^{-5}, 12 \cdot 10^{-5}, 3 \cdot 10^{-5})$. We take the particular solution $\Gamma_{\alpha,\beta}$ given by (3.10) with $(\alpha, \beta) = (\pi/2, \pi - 0.0007)$. The corresponding parameter values are: $\epsilon = 1$, $\delta = 0.15 \cdot 10^{-3}$, $b = 4320.3618$, $(g_1, g_2) = (0.499987, .494117)$.

The lower bound (3.13) writes as follows: $\tau \geq 0.137267$ and the interval in (2.36) is given by $\theta \in (-\infty, -74.8181] \cup [10.0929, +\infty)$. One checks that $(g_1, g_2) \in E$ together with (2.35) given by $\tau \in (0, 15.2680)$.

Performing the numerical integration of the differential system (3.1) one can calculate the smallest $\tau^* > 0$ such that $\sin(x_1(\tau^*)) = \sin(x_1(0)) = 0$ which corresponds to the value $\tau^* = 11.6662$. Thus, the upper and lower bounds for τ^* given by Theorem 2.3 are quite satisfactory.

The aim of our study was to elaborate on the ideas of Poincaré and Sundman [2] from the Celestial Mechanics providing converging time series solutions for the n–body problem. Unfortunately, these solutions, though converging for all values of time, have very slow convergence. One would have to sum up milliards of terms to gain any significant qualitative information above the motion of particles. The present study was designed to test the hypothesis that this gap can be overcome by replacing the power series with functional continued fractions. We notice that this important issue has not yet been

addressed fully in the literature and a number of aspects of the g–fractions approach presented here require further investigation.

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